STABILITY OF PLANE COUETTE FLOW

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More than thirty studies have been devoted to the stability of plane Couette flow. The first was an article by Kelvin [1], and the problem is still attracting the attention of investigators [2]. It was quite correctly noted in [3] that although the investigators are convinced of the stability of Couette flow to small disturbances, the numerical and asymptotic calculation regions do not overlap and therefore statements concerning stability have been of a specific nature.

Numerical analysis on an electronic computer has been carried out for selected values of the wavenumebr $\alpha$ and the Reynolds number R, with the values falling in the region $\alpha R \leqslant 10^4$ [4-6].

The objective of the present article is to: 1) extend the numerical calculation region to $\alpha R \sim 10^7$; 2) study the dependence of the eigenvalue on $\alpha$ over the entire range $0 \leq \alpha < \infty$; 3) on the basis of the numerical analysis, predict the behavior of the eigenvalues for arbitrary $\alpha$ and R; 4) join the region of numerical calculations with the region of applicability of the asymptotic methods.

The problem reduces [3] to finding the eigenvalues of the equation

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,(y - C)\,(\varphi'' - \alpha^2\varphi)\quad (-1 \leqslant y \leqslant 1) \tag{1}$$

with the boundary conditions

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0 \tag{2}$$

Here $C = X + iY$ is the unknown eigenvalue, the case $Y < 0$ corresponds to decaying disturbances.

All the parameters in (1) are dimensionless and are based on the channel halfwidth, wall velocity, and molecular viscosity.

The eigenvalues are found by the step-by-step integration method [7], modified for the case of asymmetric profiles.

A logarithmic scale, where $L = \log(1 - |X|)$, is used for clarity in illustrating the asymptotic relations in the figures.

Figure 1 shows the first six eigenvalues as a function of R for $\alpha = 1$. These calculations were made primarily as a check and for comparison with the results of other authors. For small R the eigenvalues are purely imaginary in accordance with the general relations noted in [8]. With increase of R the eigenvalues merge by pairs and after passing through the multiple point form the pairs $C = \pm X + iY$. The first two pairs of eigenvalues coincide to within the plotting error with the results of [4, 5] over the entire region investigated by these authors ($R \leqslant 10^3$). For the third pair of eigenvalues with $R > 100$ the results of the present author differ from those of Birikh even with account for the corrections [5].

This is apparently explained by the fact that the errors of the variational method increase significantly with increase of the eigenvalue number and with increase of R, while the present calculations were made with a fixed accuracy which is independent of these factors. We note also that the results obtained by numerical analysis differ considerably from the calculations of [9], in which the asymptotic method was used.

The detailed asymptotic analysis made in [10] showed that the following estimate is valid for large $\alpha R$

$$C = \pm 1 + (\alpha R)^{-1/3}\eta + 0\,[(\alpha R)^{-2/3}] \tag{3}$$

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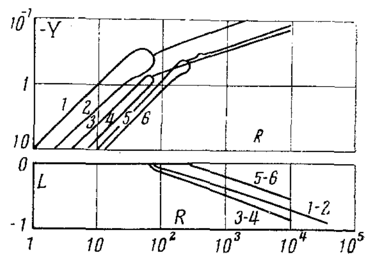

Fig. 1

We see clearly from Fig. 1 that for $R > 10^3$ the eigenvalues approach the asymptotic relations and do so in complete agreement with (3). For example, the first pair of eigenvalues is described well by the relation

$$C = \pm 1 + (\alpha R)^{-1/3} (\pm 4.1 - i1.1) \quad (4)$$

The numerical calculations were made up to R = 34,000 and were terminated because of the obvious asymptotic behavior of C. It is known [3] that the instability region may encompass a comparatively narrow range of variation of $\alpha$ and therefore for a complete stability analysis we must investigate the entire wavenumber range $0 \leq \alpha < \infty$.

In Fig. 2 the continuous curves are the results of numerical calculations of the relation $Y(\alpha)$ for R = 5, $10^2$, $10^3$, $10^4$ for the first eigenvalue (curves 1, 2, 3, 4, respectively). For small $\alpha$, just as for small R, the relation holds [8]

$$Y = -\pi^2 / \alpha R \quad (5)$$

and in this case X = 0. It is not difficult to see that for sufficiently large $\alpha$ the following relation holds:

$$Y = -\alpha / R \quad (6)$$

Relations (5) and (6) are represented by the straight lines 7 and 8 respectively in Fig. 2.

In the general case the relation $Y(\alpha)$ has two local maxima. The first is reached in the zone of monotonic disturbances (X = 0). The position of this maximum as a function of R was calculated in the range $0 \leq R \leq 10^5$ [the relations X(R) and Y(R) along the local maxima are shown in Fig. 2 by the dashed lines 5 and 6 in the region of large and small $\alpha$ respectively]. The following relation is satisfied over the entire range of R examined:

$$\Pi(R) \equiv \max_x Y \approx 4\pi / \alpha_{max} R \quad (7)$$

where $\alpha_{max}$ varies from 2.81 for the stationary fluid to zero as $R \to \infty$ and the quantity $\alpha R$ approaches the limiting value $\alpha R = 55.2$ with increase of R. Thus, in the region of monotonic distrubances $Y < -0.226$ and this is in agreement with the rigorous stability proof obtained in [11].

Then the breakpoint in the relation $Y(\alpha)$ corresponds to merging of the first two eigenvalues and generation of a pair of complex conjugate decrements. The second eigenvalue is not shown in Fig. 2, since it does not affect the stability analysis.

The second local maximum arises at the breakpoint for R = 8 and shifts in the direction of larger $\alpha$ with increase of R. The relation $\Pi(R)$ was calculated up to $\alpha R = 3 \cdot 10^7$ and for $\alpha R > 100$ the following relation holds approximately:

$$C = \pm 1 + (\alpha R)^{-1/3} (\pm 3 - i\pi^{3/2} / 3) \quad (8)$$

After reaching the second local maximum (or after passing the breakpoint for R < 8) the relation $Y(\alpha)$ rapidly approaches the asymptote (6) with the difference between the disturbance phase velocity X and the wall velocity approaching zero like

$$1 - |X| = \{^8/_3\, 2^{-\lg R} \alpha^{-2/3}\} \quad (9)$$

We see from Fig. 2 that for $R > 10^3$ the eigenvalue for $\alpha < 1$ depends only on the complex $\alpha R$. Using the numerical results obtained, we can predict the relation $C(\alpha)$ for any large values of R. For $\alpha R < 10^4$ this relation agrees with the relation for $R = 10^4$, then up to $\alpha \sim 1$, it has the form (in Fig. 2 this relation corresponds to the straight line 9)

$$C = \pm 1 + (\alpha R)^{-1/3} (\pm 4.1 - i1.1) ; \quad (10)$$

for $\alpha = 0.632$, $\sqrt{R}\, Y(\alpha)$ reaches a maximum and the value of $C(\alpha)$ is calculated using (8), then Y approaches the asymptote $Y = -\alpha / R$ and X obeys (9).

For the stability analysis it suffices to be certain that $\Pi(R) < 0$ for any R. As for the maximum located in the region of monotonicity of the disturbances, with increase of R, $\Pi(R)$ approaches the limiting values $\Pi = -0.225$, as we see from Fig. 3 (curve 2). For the maximum in the region of oscillatory

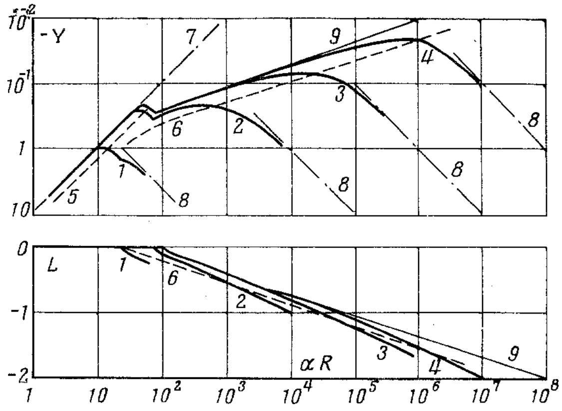


Fig. 2

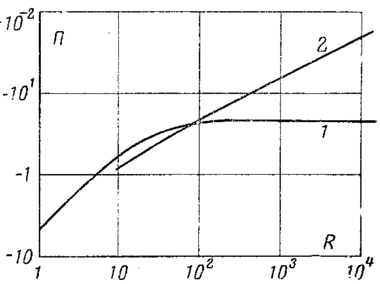


Fig. 3

disturbances (curve 2 in Fig. 3) the relation (8) predicts negativity for all R. We note that Wasow [10] proved stability for large R in the case of relation (3). The numerical calculations were made up to values of $\alpha$R for which in (3) we can neglect the last term, i.e., the regions of asymptotic and numerical analysis overlap. Thus, although there is no rigorous proof, once again the stability of Couette flow to small disturbances is shown over practically the entire range of variation of $\alpha$ and R.

In conclusion, I would like to discuss the experimental results on turbulization of Couette flow. Very few experimental studies have been made of the stability of plane Coutette flow. This is explained by the fact that it is considerably more difficult to organize the pure experiment than, for example, for flow between coaxial rotating cylinders.

Reichardt [12] indicates the range $600 < R < 1450$ in which he observed transition from the laminar to the turbulent regime. Kohlman [13] cites a broader range: $10^3 < R < 10^4$. It appears that the more careful experiments made it possible to prolong the laminar regime to higher Reynolds numbers, as is observed for the case of the circular pipe. This is pointed up by the asymptotic and numerical results indicating Couette flow stability tor any Reynolds number to sufficiently small distrubances.

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